# Algpath: A Software for Certified Algebraic Path Tracking

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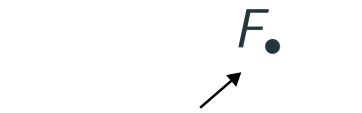
# Introduction




**Homotopy continuation**

**Input:** $F_\bullet$

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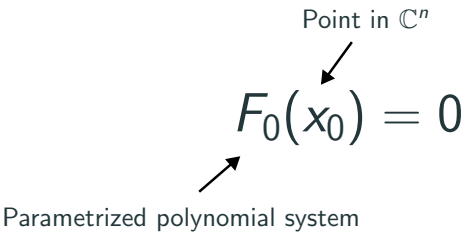


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**Input:** $F_\bullet$, $x_0$

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Unique continuous extension

$$F_t(x_t) = 0, \quad \forall t \in [0, 1]$$

Parametrized polynomial system

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**Homotopy continuation**

**Input:** $F_\bullet$, $x_0$

**Output:** $x_1$

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Algpath is a mixed precision, certified, predictor corrector loop, homotopy continuation implementation (based on interval arithmetic).

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- ⊕ Fast
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**Update**

Implements **mixed precision**.

- Wraps Arb[1] for the adaptive precision arithmetic.
- Has little overhead over the previous implementation.

[1] F. Johansson. "Arb: efficient arbitrary-precision midpoint-radius interval arithmetic"

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- ⚠ Ensure a very small overhead
- ⚠ Implement it in rust (statically typed)

# Benchmarks

| | | | HomotopyContinuation.jl[2] | | | Algpath | | |
|---|---|---|---|---|---|---|---|---|
| name | dim. | max. deg. | time (s) | failures | max. steps | time (s) | max. prec | max. steps |
| dense | 1 | 1000 | **12** | | 100 | **19 min** | 59 | 17 k |
| dense | 1 | 2000 | **50** | 3 | 93 | **2 h** | 62 | 69 k |
| katsura | 21 | 2 | **4 h** | | 469 | **96 h** | 65 | 12 k |
| resultants | 3 | 16 | **96** | | 152 | **26 h** | 75 | 5399 |
| resultants | 2 | 40 | | 200 | | **238** | 69 | 1412 |
| structured * | 3 | 10 | **3.4** | | 118 | **2.1** | 53 | 313 |
| structured * | 3 | 20 | **3.5** | 12 | 164 | **6.1** | 56 | 634 |
| structured * | 3 | 30 | **3.3** | 92 | 133 | **37** | 71 | 818 |

**Figure 1:** Total degree homotopy benchmarks. A * means that only 100 random roots were tracked.

[2] Breiding, P., Timme, S. HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia.

# Test data

We tested systems of the form $g_t(z) = tf^{\odot}(z) + (1-t)f^{\triangleright}(z)$ ($f^{\triangleright}$ is the start system, $f^{\odot}$ is the target system).

## Target systems

- Dense: $f_i^{\odot}$'s of given degree with random coefficients
- Structured: $f_i^{\odot}$'s of the form $\pm 1 + \sum_{i=1}^{\ell} \left( \sum_{j=1}^{n} a_{i,j} z_j \right)^d$, $a_{i,j} \in_R \{-1, 0, 1\}$
- Resultants: pick $h_1, h_2 \in \mathbb{C}[z_1, \cdots, z_n][y]$, compute their resultant $h \in \mathbb{C}[z_1, \cdots, z_n]$ and fill with random dense polynomials
- Katsura family (sparse - high dimension - low degree)

## Start systems

- Total degree homotopies: $f_i^{\triangleright}$'s of the form $\gamma_i(z_i^{d_i} - 1)$, $\gamma_i \in_R \mathbb{C}$, $d_i = \deg f_i^{\odot}$