Certified Algebraic Path Tracking with Algpath

Alexandre Guillemot & Pierre Lairez
MATHEXP, Université Paris–Saclay, Inria, France

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Certified path tracking

F

Parametrized polynomial system

Certified homotopy continuation

Input: F

Certified path tracking

Point in $\mathbb{C}^n$

$$F_0(\zeta_0) = 0$$

Parametrized polynomial system

Certified homotopy continuation

Input: F, ζ_0

Certified path tracking

Unique continuous extension

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Parametrized polynomial system

Certified homotopy continuation

Input: F, ζ_0

Output: A "certified approximation" of ζ

Related work

Noncertified path trackers

- PHCpack by Verschelde (1999)
- Bertini by Bates, Sommese, Hauenstein, and Wampler (2013)
- HomotopyContinuation.jl by Breiding and Timme (2018)

Certified path trackers using Smale's alpha-theory

- NAG for M2 by Beltrán and Leykin (2012, 2013)

Certified path trackers in one variable

- SIROCCO by Marco-Buzunariz and Rodríguez (2016)
- Kranich (2016)
- Xu, Burr, and Yap (2018)

Certified path trackers using interval arithmetic

- Kearfott and Xing (1994)
- van der Hoeven (2015) *Krawczyk operator + Taylor models*
- Duff and Lee (2024)

Algpath

Features

- Rust implementation available at **https://gitlab.inria.fr/numag/algpath,**
- **certified** corrector-predictor loop,
- relies on **interval arithmetic** and **Krawczyk's method**,
- **SIMD double precision interval arithmetic** following [Lambov, 2008],
- NEW! **adaptive precision** using **Arb**[1],
- NEW! **mixed precision** between double precision and Arb **without overhead**.

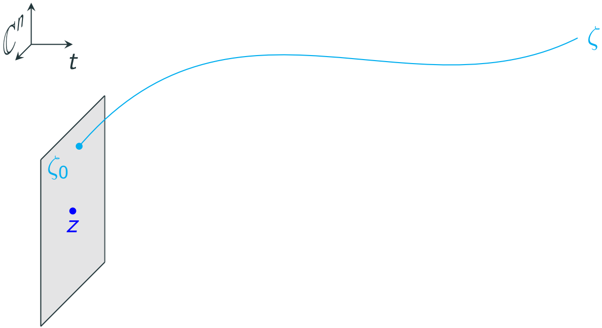
Applications

- Monodromy computations,
- **Braid computations**

[1] F. Johansson. "Arb: efficient arbitrary-precision midpoint-radius interval arithmetic"

Certified corrector-predictor loop

Recall: for all $t \in [0,1]$, $F_t(\zeta_t) = 0$




```
def track(F, z):
    t ← 0;    L ← []
    while t < 1:
        z ← refine(F_t, z)
        δ ← validate(F, t, z)
        t ← t + δ
        append (t, z) to L
    return L
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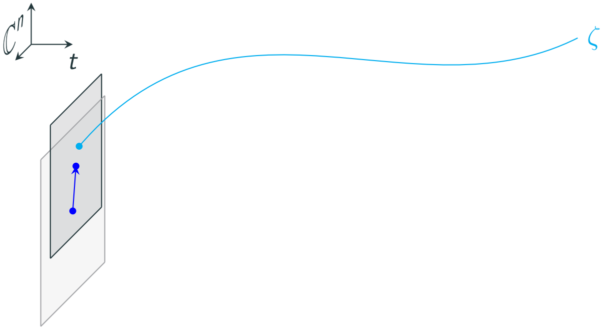

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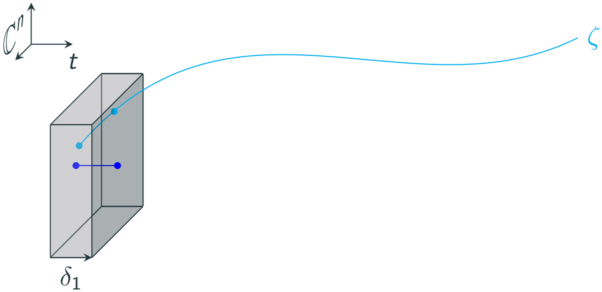

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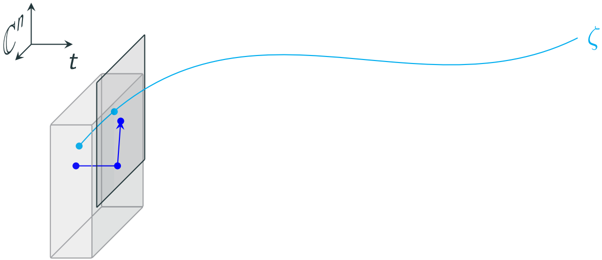

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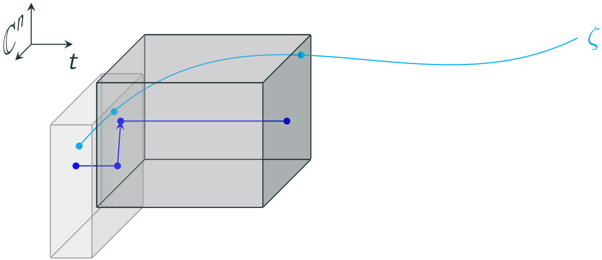

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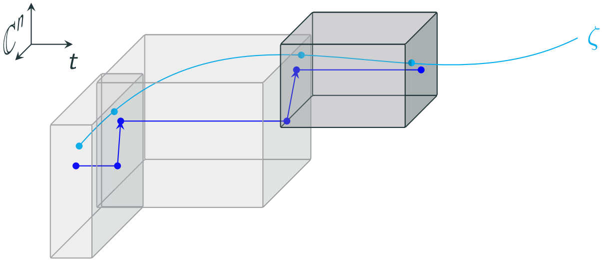

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- Precision is managed globally
- A change of precision induces **no changes on data, only operations are changed**
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In practice we use Arb and decrease precision by 1 bit at each iteration of the main loop.

Mixed precision

Double precision SIMD interval arithmetic is faster than Arb, but it lacks the ability to manage precision...

Goal

Use double precision when possible, else use Arb. We want to have no overhead over double precision only.

- 💡 Data can either be double precision or Arb balls. Operations manage arithmetic switch depending on precision
- ! Overhead
- ! Challenging implementation

```
enum MixedRI {
  Fast(F64RI),
  Accurate(Arb),
}
```

Spacing arithmetic switches

One iteration of the main loop

```
def one_step(F, m):  # m isolating box
    try:
        convert m to double precision
        perform a corrector-predictor round at double precision
    except:
        convert m to Arb  # ⚠ exact interval conversions are tricky
        perform a corrector-predictor round using Arb
```

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			algpath	algpath (fixed precision)
name	dim.	max deg	time (s)	time (s)
dense	1	100	0.4	0.4
katsura	16	2	42 min	41 min
dense	2	50	588	588

Conclusion

			HomotopyContinuation.jl			algpath		
name	dim.	max deg	time (s)	fail.	max.	time (s)	prec.	max.
dense	1	1000	**6.8**		100	**12 min**	59	17 k
dense	1	2000	**26**	**3**	79	**1 h**	62	69 k
katsura	21	2	**4 h**		468	**60 h**	65	12 k
resultants	3	16	**5.6**		128	**92**	58	1857
resultants	2	40		**200**		**185**	69	1414
structured *	3	10	**3.0**		118	**1.5**	53	313
structured *	3	20	**3.0**	**12**	164	**4.2**	56	634
structured *	3	30	**2.9**	**92**	133	**24**	71	818

Figure 1: Total degree homotopy benchmarks. A * means that only 100 random roots were tracked.

[2] Breiding, P., Timme, S. HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia.

Test data

We tested systems of the form $g_t(z) = tf^{\odot}(z) + (1-t)f^{\triangleright}(z)$ ($f^{\triangleright}$ is the start system, $f^{\odot}$ is the target system).

Target systems

- Dense: $f_i^{\odot}$'s of given degree with random coefficients
- Structured: $f_i^{\odot}$'s of the form $\pm 1 + \sum_{i=1}^{\ell} \left(\sum_{j=1}^{n} a_{i,j} z_j \right)^d$, $a_{i,j} \in_R \{-1, 0, 1\}$
- Resultants: pick $h_1, h_2 \in \mathbb{C}[z_1, \cdots, z_n][y]$, compute their resultant $h \in \mathbb{C}[z_1, \cdots, z_n]$ and fill with random dense polynomials
- Katsura family (sparse - high dimension - low degree)

Start systems

- Total degree homotopies: $f_i^{\triangleright}$'s of the form $\gamma_i(z_i^{d_i} - 1)$, $\gamma_i \in_R \mathbb{C}$, $d_i = \deg f_i^{\odot}$

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